8.2 Step Functions

**Step functions** are piecewise functions that produce graphs that look like stair steps. They reduce any number within a given interval into a single number. Typically parking garages, boat rentals, or any place that charges per segment of an hour, is actually using a step function rather than a linear function.

**Overview of Step Function Notation**

There are two main step functions that we will be using: the floor function and the ceiling function. The floor function $f(x) = \lfloor x \rfloor$ will output the greatest integer that is less than or equal to the input. For example, $\lfloor 7.2 \rfloor = 7$ or $\lfloor -2.2 \rfloor = -3$. Just like the notation has the extra lines below, we always take the input values down to the nearest integer.

The ceiling function $f(x) = \lceil x \rceil$ will output the least integer that is greater than or equal to the input. For example, $\lceil 7.2 \rceil = 8$ or $\lceil -2.2 \rceil = -2$. Just like the notation has the extra lines above, we always take the input values up to the nearest integer.

**Graphing Step Functions**

Let’s start by evaluating and graphing the parent floor function, $f(x) = \lfloor x \rfloor$. We’ll make an $x/y$ chart and plot the points on the coordinate plane as follows:

<table>
<thead>
<tr>
<th>$x$</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Notice that if we were to graph these points, we might think it was a linear function. To see the “steps” appear, let’s think of some values in between the integer inputs as follows.

<table>
<thead>
<tr>
<th>$x$</th>
<th>-1.8</th>
<th>-1.2</th>
<th>-0.5</th>
<th>-0.3</th>
<th>0.1</th>
<th>0.4</th>
<th>1.6</th>
<th>1.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>-2</td>
<td>-2</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Now we can see why it is called a step function. Any input in the domain $-2 \leq x < -1$ will be assigned to an output of $-2$. That’s what leads us to the open and closed circles as well. Since an input of $-2$ gives the output of $-2$, we use a closed circle there. However, on the other end, an input of $-1$ will not give us an output of $-2$. So that is an open circle.

Next let’s graph a couple more step functions, to make sure we have the hang of it.
\[ g(x) = [x + 4] \]

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>

\[ h(x) = [x] + 2 \]

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(x) )</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

Notice that the ceiling function leads to open circles on the other end. This is because if we input an integer value, it will stay the same, but as soon as get any bigger, it will bump up to the next integer. Both the floor and ceiling functions can be translated (or shifted). So let's examine these transformations in more detail.

### Transforming Step Functions

The general form of a step function is \( f(x) = a\lfloor bx - h \rfloor + k \). As you might already be able to guess, the \( h \) value controls the shift left and right while the \( k \) value controls the shift up and down. The \( a \) and \( b \) values are perhaps a bit more perplexing. Let's look at those a bit closer.

\[ f(x) = 3[x] \]

Notice that the \( a \) controlled the "height" of the steps. This should make sense because we are multiplying the output by the value \( a \), so when \( a \) gets bigger, the output is bigger. When \( a \) gets smaller, the output is smaller.
(Sorry about zooming in so far on the graph, but graphing in Word is annoying.) This also relates to our knowledge of transformations that $a \ast f(x)$ moves the function $a$ times farther from the $x$-axis.

Since the $a$ value controlled the height of the steps, what would you guess the $b$ value controls? Let’s find out by looking at the following graphs. Note we’re also switching to the ceiling function so the open circles will now be on the left.

$$f(x) = [3x]$$

$$g(x) = \left\lceil \frac{1}{3}x \right\rceil$$

Notice that the $b$ controlled the “length” of the steps. This should make sense because we multiplying the input by the value $b$ before performing the step function. So when $b$ gets bigger, the steps get smaller. When $b$ gets smaller, the steps get bigger. This again relates to our knowledge that $f(bx)$ will pull the function $b$ times closer to the $y$-axis.

**Solving Step Equations**

To solve step equations, we must think about the inequality that they represent. For example, if we had the equation $[x] = 5$, then we know that $x \geq 5$ and $x < 6$. Any value in that interval will turn into five with the floor function. Similarly, $[x] = 7$ occurs when $x > 6$ and $x \leq 7$. Thinking of those two inequalities, when can then solve any step function.
Let’s consider solving the equation $2[x + 3] - 4 = 6$. We can use inverse operations until we get it down to the ceiling function as follows:

$$2[x + 3] - 4 = 6$$
$$2[x + 3] - 4 + 4 = 6 + 4$$
$$2[x + 3] = 10$$
$$\frac{2[x + 3]}{2} = \frac{10}{2}$$
$$[x + 3] = 5$$

Now that we are down to the step function alone, we know that $x + 3$ must be greater than 4 (to round up to 5) but no bigger than 5 itself. This is because the ceiling function will round the value up to the nearest integer. That gives us two inequalities to solve:

Lower Bound:
- $x + 3 > 4$
- $x + 3 - 3 > 4 - 3$
- $x > 1$

Upper Bound:
- $x + 3 \leq 5$
- $x + 3 - 3 \leq 5 - 3$
- $x \leq 2$

Now we have our final solution which we could write as $1 < x \leq 2$ and even plot on a number line if we so desired.

**Real-Life Step Equations**

Let’s say you travel to a lake and want to rent a paddle boat. They charge $7 for every 15 minutes plus a flat fee of $25 for insurance. In the past, we modeled this situation with a linear function of $c(m) = \frac{7}{15}m + 25$, but the boat rental place doesn’t actually charge for each minute. If you have the boat five or ten minutes it costs the same as if you kept the boat out the full fifteen minutes. To more accurately model this situation, we could use the equation: $c(m) = 7\left\lfloor \frac{1}{15}m \right\rfloor + 25$.

Taxes, rentals, stamps, and many other situations we would typically think of as being modeled by a linear function are actually step functions (and usually ceiling functions since they round the cost up instead of down).
Lesson 8.2

Fill out an $x/y$ chart and graph each of the following functions.

1. $f(x) = \frac{1}{2} \lfloor x \rfloor$
2. $g(x) = -[x + 2]$
3. $h(x) = [2x] - 4$

4. $f(x) = \left\lfloor \frac{1}{2} x \right\rfloor + 3$
5. $g(x) = [-x] - 4$
6. $h(x) = -\frac{1}{3} [x - 1]$

Given that $f(x) = |x|$ is the parent function, describe the transformation of $g(x)$.

7. $g(x) = \left\lfloor \frac{1}{2} x + 2 \right\rfloor$
8. $g(x) = -|x| + 4$
9. $g(x) = |x - 6|$
Given that \( f(x) = \lfloor x \rfloor \) is the parent function, describe the transformation of \( g(x) \).

10. \( g(x) = 2\lfloor x \rfloor - 7 \)  
11. \( g(x) = -\lceil 4x \rceil \)  
12. \( g(x) = -x + 1 + 5 \)

Solve the following equations.

13. \( 2\lfloor x \rfloor = 10 \)  
14. \( [x + 4] + 5 = 4 \)  
15. \( \left\lfloor \frac{1}{2} x - 1 \right\rfloor + 4 = 0 \)

16. \( [-4x] - 9 = -5 \)  
17. \( -[x - 5] = 15 \)  
18. \( \left\lceil \frac{1}{3} x + \frac{1}{2} \right\rceil = 8 \)

Create a function to model the following situations.

19. You want to bring cupcakes to school for your birthday. Each case comes with 12 cupcakes and costs $6.95. Create a function that models the cost in terms of the number of students in your class.

20. Renting jet skis in the Bahamas costs $40 per hour plus a $15 gas fee. Create a function that models the cost in terms of the number of hours the jet skis were rented.

21. Laser tag at Fred’s Family Fun costs $6 for every segment of 15 minutes of play, plus a $5 battery fee. Create a function that models the cost in terms of the number of minutes playing tag.

22. A textbook company charges $725 for each case of books that it sells. A case can contain any number of books up to 30 books. They charge a flat shipping fee of $100. Create a function that models the cost in terms of the number of books needed.

23. Long distance phone calls cost $0.99 for the first minute, and $0.39 for every minute after that. Create a function that models the cost in terms of the duration of the phone call in minutes.

24. You’re ordering pizza for your birthday party. You estimate that each pizza will serve 4 people. Create a function that models the number of pizzas you need to order in terms of the number of people attending.